String Perturbation Theory

In a recent Letter[1] Gross and Periwal reported that they have suceeded in proving that the string theoretic perturbation expansion for the partition function diverges at least as fast as $\Sigma h!(g^2)^h$. Not only that, but in addition they have reported that they have proven that every series term is positive. Their result is a significant one, because as they rightly point out "if perturbative string theory were to make sense, string theory would have nothing to do with physical reality, since there are many features of all perturbative treatments of string theory that that are not shared by the real world." From these results, they unfortunately overly hastily conclude that the string theory perturbation expansion is not Borel summable.

Without regard as to whether string perturbation theory is or is not in fact summable, it is possible to see from a simple example that the properties reported by Gross and Periwal do not suffice to prove Borel non-summability, although if our example were the correct string theory function, a complex result for the sum would appear. Consider the example,

$$E(g) = \Sigma_0^\infty h! g^{2h} = \int_0^\infty \frac{e^{-t} dt}{1 - g^2 t}. \tag{1}$$

This function (due to Euler) obeys the divergence and positivity properties shown by Gross and Periwal and yet is summed perfectly well for $g^2 < 0$. Now it was objected that the series was not summable for positive real g^2, and at first glance, the above representation would seem to make no sense because of the singularity in the integrand for this case. However, by Cauchy's theorem and the fact that e^{-z} goes to zero as $|z|$ goes to ∞ with $|\arg(z)| < \frac{\pi}{2}$ the contour of integration can be rotated to run along any ray in the right half-plane. Therefore, $E(g)$ can be re-expressed as

$$E(g) = \int_0^\infty \frac{\exp(-\tau\cos\theta)[\cos(\tau\sin\theta) + i\sin(\tau\sin\theta)]e^{i\theta} d\tau}{1 - g^2 \tau e^{i\theta}}. \tag{2}$$

In this form ($\theta < 0$) we can analytically continue from g^2 real and negative through the upper half-plane to g^2 real and positive. Thus our example is Borel summable to a complex result on the positive real axis and of course also to its complex conjugate by taking $\theta > 0$ and analytically continuing through the lower half-plane.

More generally, of course, it is not necessary to analytically continue the function to the second Riemann sheet. If for example the singularies of the Borel transform all lie in the right half-plane, as is the case, *a fortiori*, for the leading singularity which lies on the positive real axis by the positivity reported by Gross and Periwal, then the representation of eq. (2) allows the extension of the definition of the function to a function analytic in the cut complex plane $(0,+\infty)$. By taking a limit the values on the positive real axis can be defined as the boundary values of an analytic function. These sorts of ideas are familiar in physics from the study of dispersion relations[2] in, for example, the cut energy-plane. For a more general discussion of Borel summation a classic reference is Hardy.[3] In addition the extentions of Watson's theorem due to Graffi *et al.*[4] and Sokal[5] are worth mention.

In case the string perturbation theory should prove to be summable, and a reasonable number of terms in the series expansion be computed, then a possible approach to its approximate summation would be the use of the Padé-Borel summation method.[4,6] The analysis of the values on the positive real axis implied by this method follows directly from that given here for our example.

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George A. Baker, Jr.

Theoretical Division, Los Alamos National Laboratory

University of California, Los Alamos, N. M. 87545, USA

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